# Asymptotic winding angle distributions for planar Brownian motion with drift



1993 J. Phys. A: Math. Gen. 26 5637

(http://iopscience.iop.org/0305-4470/26/21/005)

J. Phys. A: Math. Gen. 26 (1993) 5637–5643. Printed in the UK

# Asymptotic winding angle distributions for planar Brownian motion with drift

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Received 20 May 1993

**Abstract.** We study the asymptotic winding angle distribution of a two-dimensional Brownian curve around a given point. The influence of boundaries and/or drifts on this distribution is addressed. In particular, we show that, for some special drifts, the limiting distribution is given in terms of Lévy laws.

The study of winding properties of Brownian curves goes back to the pioneering work of Lévy [1] and Spitzer [2]. Since that time, some effort has been devoted to these problems by mathematicians [3, 4] as well as by physicists [5–7]. A variety of methods have been devised ranging from the use of refined probability techniques to a formulation involving constrained path integrals or two-dimensional conformal field theory [8].

From a physical point of view, it is clear that entanglements of polymer chains in network systems or in concentrated solutions may play an important role, especially concerning their elastic or viscoelastic properties. The archetypal problem is to consider entanglements of a macromolecule with a straight line [9]. Neglecting excluded volume effects, the macromolecule can in some cases be modelled by a free three-dimensional random walk. In the continuum limit (diffusion approximation), using the factorization property of Brownian motion, one is left with a two-dimensional problem, namely the study of the angle $\theta(t)$ wound around a given point $O$ by a planar Brownian curve of length $t$. This problem was first addressed by Spitzer in 1958 who proved that the rescaled angle $\theta' = 2\theta/\ln t$ is distributed according to the Cauchy law

$$P(\theta') = \frac{1}{\pi(1+\theta'^2)} \tag{1}$$

whose characteristic function

$$K(\lambda') = \langle \mathrm{e}^{\mathrm{i}\lambda'\theta'} \rangle = \mathrm{e}^{-|\lambda'|} \tag{2}$$

gives infinite even moments of all orders.

Pitman and Yor [3] extended this result by splitting the total winding angle $\theta$ into $\theta_1$ and $\theta_2$, called respectively the small and big winding angles. $\theta_1(\theta_2)$ is the total angle wound by the Brownian particle when it moves inside (outside) the disc of centre $O$ and radius $R$ (see figure 1). They obtained the characteristic function $K(\lambda'_1, \lambda'_2)$ of the joint limit law $P(\theta'_1, \theta'_2)$

$$K(\lambda'_1, \lambda'_2) = \frac{1}{\mathrm{ch}\,\lambda'_2 + (|\lambda'_1|/\lambda'_2)\mathrm{sh}\,\lambda'_2} \tag{3}$$

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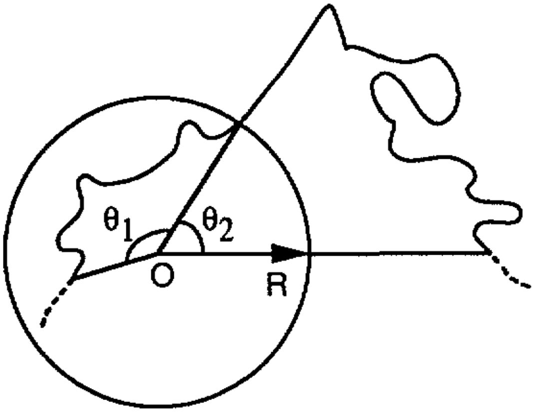


**Figure 1.** A Brownian particle entering into the disc of radius $R$. $\theta_2$ is the total angle wound by the particle when it is outside the disc (big winding). $\theta_1$ is the total angle wound by the particle when it is inside the disc (small winding).

where $\theta_1' = 2\theta_1/\ln t$, $\theta_2' = 2\theta_2/\ln t$.

More recently, Belisle [4] obtained an analogue of Spitzer's law for windings of random walk on a square lattice. The resulting expression

$$P(\theta') = \frac{1}{2\mathrm{ch}\,(\pi\theta'/2)} \Rightarrow K(\lambda') = \frac{1}{\mathrm{ch}\,\lambda'} \tag{4}$$

leads to the same law as (3) for the big windings only, i.e. when $\lambda_1' = 0$.

The occurence of infinite moments in Spitzer's law was also analysed by Rudnick and Hu [6]. By removing from the plane a disc around $O$, they proved that the tail of the probability distribution reads, when $|\theta'| \to +\infty$,

$$\lim P(\theta') \propto \mathrm{e}^{-\pi|\theta'|/2}\,. \tag{5}$$

In contrast with (2), all positive moments are now finite. Thus, the behaviour of the Brownian particle in the vicinity of $O$ plays an essential role in the construction of the law.

The purpose of this work is to investigate this feature in detail by considering Brownian motions with drift (and/or excluded area) on a finite domain. In particular, it will be shown that limiting distributions can take the form of generalized Levy laws for some special drifts. Interesting physical consequences of the presence of a drift on the winding properties of a polymer chain have already been discussed by Houchmandzadeh *et al* [7]. These authors have studied the influence of the interaction of the polymer with a rigid rod on the localization transition. Using a path-integral formulation, we will show along similar lines that the asymptotic law is governed by the non-analytic behaviour of the ground-state energy of an associated quantum mechanical Hamiltonian.

To begin, let us consider a Brownian particle starting at $\boldsymbol{r}_0$ (at time $t_0 = 0$) and arriving at $\boldsymbol{r}$ (at time $t$) with a probability $P \equiv P(\boldsymbol{r}, \boldsymbol{r}_0, t)$. The particle is subjected to a force $\boldsymbol{F} = -\boldsymbol{\nabla}\mathcal{U}$ depending on the position. The corresponding Langevin equation reads

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{F} + \eta(t) \tag{6}$$

where $\beta(t)$ is a Gaussian white noise such that

$$\langle \eta_\alpha(t)\rangle = 0 \qquad \langle \eta_\alpha(t)\eta_\beta(t')\rangle = 2D_0\delta_{\alpha\beta}\delta(t-t')\,.$$

The time evolution of $P$ is therefore described by the Fokker–Planck equation

$$\frac{\partial P}{\partial t} = \nabla(D_0\nabla P - \boldsymbol{F}P)\,. \tag{7}$$

Without loss of generality, we take $D_0 = 1$ for the diffusion constant.

In the following we will only consider radial forces, therefore the potential $\mathcal{U}$ only depends on the radial distance $r$.

It is convenient to set

$$P = \exp\left(-\frac{\mathcal{U}}{2}\right)\chi\,. \tag{8}$$

Then (7) leads to

$$-\frac{\partial\chi}{\partial t} = H\chi \tag{9}$$

where $H = -\Delta + V = -\Delta - \frac{1}{2}\Delta\mathcal{U} + \frac{1}{4}(\nabla\mathcal{U})^2$. The probability $P$ can thus be expressed in terms of the eigenfunctions $\psi_n$ and eigenvalues $E_n$ of the above Hamiltonian:

$$P(\boldsymbol{r},\boldsymbol{r}_0,t) = \exp-\left(\frac{\mathcal{U}(\boldsymbol{r})-\mathcal{U}(\boldsymbol{r}_0)}{2}\right)\sum_n \psi_n^*(\boldsymbol{r})\psi_n(\boldsymbol{r}_0)\,\mathrm{e}^{-E_nt}\,. \tag{10}$$

The expression of the ground-state wavefunction ($E_0 = 0$)

$$\psi_0(\boldsymbol{r}) = \exp-\frac{\mathcal{U}(\boldsymbol{r})}{2} \tag{11}$$

shows that $P$ is properly normalized: $\int \mathrm{d}^2\boldsymbol{r}\,P(\boldsymbol{r},\boldsymbol{r}_0,t) = 1$.

In a path-integral formulation $P$ reads

$$P(\boldsymbol{r},\boldsymbol{r}_0,t) = \exp\left[-\left(\frac{\mathcal{U}(\boldsymbol{r})-\mathcal{U}(\boldsymbol{r}_0)}{2}\right)\right]\int_{\boldsymbol{r}(0)=\boldsymbol{r}_0}^{\boldsymbol{r}(t)=\boldsymbol{r}}\mathcal{D}\boldsymbol{r}(\tau)\exp-\int_0^t\left(\frac{\dot{\boldsymbol{r}}^2(\tau)}{4}+V(\boldsymbol{r}(\tau))\right)\mathrm{d}\tau \tag{12}$$

where $\mathcal{D}\boldsymbol{r}(\tau)$ is the measure on the functional space of trajectories $\boldsymbol{r}(\tau)$.

We are interested in winding angle distributions, the final point $\boldsymbol{r}$ being left unspecified:

$$P(\theta,\boldsymbol{r}_0;t) = \int \mathrm{d}^2\boldsymbol{r}\,\exp\left[-\left(\frac{\mathcal{U}(\boldsymbol{r})-\mathcal{U}(\boldsymbol{r}_0)}{2}\right)\right]$$
$$\times\int_{\boldsymbol{r}(0)=\boldsymbol{r}_0}^{\boldsymbol{r}(t)=\boldsymbol{r}}\mathcal{D}\boldsymbol{r}(\tau)\exp-\left\{\int_0^t\left(\frac{\dot{\boldsymbol{r}}^2(\tau)}{4}+V(r)\right)\mathrm{d}\tau\right\}\delta\left(\theta-\int_0^t\dot{\theta}\,\mathrm{d}\tau\right). \tag{13}$$

A convenient way to impose the constraint in this path integral is to set

$$\delta\left(\theta-\int_0^t\dot{\theta}\,d\tau\right) = \left(\frac{1}{2\pi}\right)\int_{-\infty}^{+\infty}\mathrm{d}\lambda\,\mathrm{e}^{-\mathrm{i}\lambda(\theta-\int_0^t\dot{\theta}\,\mathrm{d}\tau)}\,.$$

The characteristic function for the winding angle at the time $t$ is therefore

$$
\begin{aligned}
K(\lambda, \boldsymbol{r}_0; t) &= \int \mathrm{d}^2\boldsymbol{r} \exp\left[-\left(\frac{\mathcal{U}(\boldsymbol{r}) - \mathcal{U}(\boldsymbol{r}_0)}{2}\right)\right] \\
&\quad \times \int_{\boldsymbol{r}(0)=\boldsymbol{r}_0}^{\boldsymbol{r}(t)=\boldsymbol{r}} \mathcal{D}\boldsymbol{r}(\tau) \exp -\left\{\int_0^t \left(\frac{\dot{\boldsymbol{r}}^2(\tau)}{4} + V(r) - \mathrm{i}\lambda\dot{\theta}(\tau)\right)\mathrm{d}\tau\right\} \\
&\equiv \int \mathrm{d}^2\boldsymbol{r} \exp\left[-\left(\frac{\mathcal{U}(\boldsymbol{r}) - \mathcal{U}(\boldsymbol{r}_0)}{2}\right)\right] \langle \boldsymbol{r}|\mathrm{e}^{-tH_\lambda}|\boldsymbol{r}_0\rangle
\end{aligned} \tag{14}
$$

where

$$H_\lambda = -\frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}(-\mathrm{i}\partial_\theta - \lambda)^2 + V(r)\,.$$

Due to the integration over $\boldsymbol{r}$, only 0-angular momentum eigenstates will contribute. Dropping the angular part of $H_\lambda$, we get

$$
\begin{aligned}
H_\lambda &= -\frac{1}{r}\partial_r(r\partial_r) + V(r) + \frac{\lambda^2}{r^2} \\
&\equiv H_0 + \frac{\lambda^2}{r^2}\,.
\end{aligned} \tag{15}
$$

Suppose the particle is allowed to wander everywhere in the plane without drift ($U(r) = V(r) = 0$); then (14) and (15) lead to

$$
\begin{aligned}
K(\lambda, \boldsymbol{r}_0; t) &= \int_0^\infty r\,\mathrm{d}r \int_0^\infty k\,\mathrm{d}k\, \mathrm{e}^{-tk^2} J_{|\lambda|}(kr) J_{|\lambda|}(kr_0) \\
&= 2^{-|\lambda|}\frac{\Gamma(|\lambda|/2+1)}{\Gamma(|\lambda|+1)}\left(\frac{r_0^2}{t}\right)^{|\lambda|/2} F\left(\frac{|\lambda|}{2}, |\lambda|+1, -\frac{r_0^2}{4t}\right)
\end{aligned} \tag{16}
$$

where $J_{|\lambda|}(kr)$ is a Bessel function of the first kind and $F$ is the confluent hypergeometric function regular at the origin. Henceforth, we will only consider the limit $t \to \infty$, $\theta \to \infty$, i.e. $r_0^2/t$ and $\lambda$ small (since $\lambda$ is the conjugate variable with respect to $\theta$, by Fourier transformation only small $\lambda$ values will give significant contributions to $P(\theta, \boldsymbol{r}_0; t)$). The asymptotic expression of $K$ is then reduced to

$$K(\lambda, \boldsymbol{r}_0; t) = \left(\frac{r_0^2}{t}\right)^{|\lambda|/2} \sim t^{-|\lambda|/2}\,. \tag{17}$$

Fourier transforming with respect to $\lambda$ we recover Spitzer's law (1).

However, the picture is somewhat different when the particle is constrained to move inside a disc of centre $O$ and radius $R$. The spectrum of $H_\lambda$ is now discrete and, in the limit $t \to \infty$, the leading part of $K$ is given by the ground state of $H_\lambda$. In this case, only the lowest-order contribution in $\lambda$, $\Delta E(\lambda) = E(\lambda) - E_0$ of the ground-state energy shift will be needed:

$$K(\lambda, r_0; t) \sim \mathrm{e}^{-t\Delta E(\lambda)}\,. \tag{18}$$

Assuming that $\Delta E(\lambda)$ can be obtained by perturbation theory, we get from (15)

$$
\begin{aligned}
\Delta E(\lambda) &= \lambda^2\left\langle \psi_0 \left| \frac{1}{r^2} \right| \psi_0 \right\rangle \\
&\equiv 2\pi\lambda^2 \int_0^R [\psi_0(r)]^2 \frac{\mathrm{d}r}{r}\,.
\end{aligned} \tag{19}
$$

If a disc of radius $R_1$ around $O$ is removed ($\psi_o(r) = 0$ if $R_1 < r < R$) then the integral exists and

$$\Delta E(\lambda) \sim \lambda^2 . \tag{20}$$

Fourier transformation of (18) shows that the random variable $\theta/\sqrt{t}$ has a Gaussian distribution; in other words the particle performs a quasi-one-dimensional Brownian motion along the tangential direction.

It is worth noticing that, qualitatively, the same result would hold for a free particle on the whole disk if

(i) $V(r) = \alpha^2/r^2$. In that case, $\psi_0(r) \sim r^{|\alpha|}$ for small $r$ and (19) exists. (Of course, (19) still exists for $V(r) = \alpha^2/r^a$, $a > 2$).

(ii) $V(r) = 0$ and only big windings are taken into account ($\theta$ is counted only when $r > R_1$, $H_\lambda \equiv H_0$ when $r < R_1$).

Consider now a particle allowed to wander everywhere inside the disc, without drift ($V(r) = 0$, $\theta$ is the total winding angle). Then $\psi_0(0) \neq 0$ and (19) diverges. This clearly shows that the leading order of the perturbation is less than 2. To proceed further, one can use Dirichlet boundary conditions: $\psi(R) = 0$. This amounts to considering Brownian curves that never meet the frontier of the disc.

The ground-state energy shift $\Delta E(\lambda)$ is obtained by comparing the boundary conditions

$$J_{|\lambda|}(kR) = 0 \tag{21}$$

and

$$J_0(k_0 R) = 0 . \tag{22}$$

For small $\lambda$ we get

$$\Delta E(\lambda) = \frac{2|\lambda|}{R^2 [J_1(s_1)]^2} \tag{23}$$

where $s_1 = k_0 R$ is the first positive zero of $J_0$. This shows that the scaling variable $\mu S\theta/t$ (where $S$ is the area of the disc and $\mu = J_1(s_1)^2/2\pi$) is distributed according to a Cauchy law.

This result can be easily recovered by a perturbative analysis. A standar perturbative treatment of the $\lambda^2/r^2$ term would, however, yield an infinite energy shift because the unperturbed wavefunction does not vanish at $r = 0$. Since the singular nature of the interaction term forces the true wavefunction to vanish at $r = 0$, it is convenient to define an auxiliary function $\psi_\lambda$ as $\psi_\lambda(r) = r^{|\lambda|}\phi_\lambda(r)$. The eigenvalue equation for $\phi_\lambda(r)$ then reads $H'_\lambda \phi_\lambda = E\phi_\lambda$, where

$$H'_\lambda = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{2|\lambda|}{r}\frac{\partial}{\partial r} . \tag{24}$$

The perturbative analysis can now be carried out. One obtains

$$\Delta E(\lambda) \sim \left\langle \psi_0 \left| -\frac{2|\lambda|}{r}\frac{\partial}{\partial r} \right| \psi_0 \right\rangle = |\lambda| \psi_0^2(0) \tag{25}$$

where $\psi_0(r) = \mathcal{N} J_0(kr)$ is the normalized ground-state wavefunction of $H_0$ on the disc. A straightforward calculation gives back (23). Similar results would be obtained with Neumann boundary conditions (i.e. pure reflection at $r = R$).

The previous analysis shows that $\theta/t$ will be a Cauchy variable whenever $\psi_0(0)$ is finite and non-zero (for instance, for a repulsive potential $V(r) = \alpha^2/r^a$, $a < 2$). The same conclusion will also hold if we only count the small windings ($r < R_1$). It is easy to prove, along the same lines, that small and big windings are not correlated in the limit $t \to \infty$. A correlation only occurs if the particle is allowed to explore the whole plane (see (3)).

However, this is not the end of the story. It can happen that (19) diverges and (25) vanishes. This implies that the critical exponent of the perturbation

$$\Delta E(\lambda) = \lambda^\alpha$$

is such that $1 < \alpha < 2$.

As an illustration consider the case

$$\psi_0(r) \sim \left[ -\ln\left(\frac{r}{R_0}\right)\right]^{-\beta} \tag{26}$$

where $0 < \beta \leqslant \frac{1}{2}$, and $R_0$ is a suitable length scale such that $R_0 > R$. The corresponding potential reads

$$V(r) = \frac{\alpha}{r^2(\ln(r/R_0))^2} \tag{27}$$

with $0 < \alpha \equiv \beta(\beta+1) \leqslant \frac{3}{4}$. One has

$$\mathcal{U}(r) = 2\beta \ln(-\ln(r/R_0))\,. \tag{28}$$

In this case it is interesting to investigate in some details the perturbation expansion; $\Delta E(\lambda)$ can be written as

$$\Delta E(\lambda) = \frac{\langle\psi_0|H_\lambda - H_0|\psi_\lambda\rangle}{\langle\psi_0|\psi_\lambda\rangle} = \frac{\int_0^R \psi_0^*(r)(\lambda^2/r^2)\psi_\lambda(r)2\pi r\,\mathrm{d}r}{\int_0^R \psi_0^*(r)\psi_\lambda(r)2\pi r\,\mathrm{d}r}\,. \tag{29}$$

Moreover, we observe that, to lowest order in $\lambda$,

$$\int_0^R \psi_0^*(r)\psi_\lambda(r)2\pi r\,\mathrm{d}r \sim \int_0^R [\psi_0(r)]^2\, 2\pi r\,\mathrm{d}r = 1\,.$$

For small $r$, we get

$$\psi_0(r) = c(-\ln(r/R_0))^{\frac{1}{2}-\sqrt{\alpha+\frac{1}{4}}} \tag{30}$$

$$\psi_\lambda(r) = c'|\lambda|^{\sqrt{\alpha+\frac{1}{4}}}(-\ln(r/R_0))^{\frac{1}{2}} K_{\sqrt{\alpha+\frac{1}{4}}}(-|\lambda|\ln(r/R_0)) \tag{31}$$

where $K_\mu$ is a modified Bessel function; $c$ is a normalization constant; and

$$c' = c \times 2^{1-\sqrt{\alpha+\frac{1}{4}}} \sin\left(\pi\sqrt{\alpha+\tfrac{1}{4}}\right)\Gamma\left(1-\sqrt{\alpha+\tfrac{1}{4}}\right)/\pi\,.$$

One has

$$\psi_\lambda(r) \underset{\lambda\to 0}{\to} \psi_0(r) \tag{32}$$

and $\psi_\lambda(r) \sim r^{|\lambda|}$ when $r \to 0$.

For some small value of $r$, say $r'$, potential $V(r)$ crosses $\lambda^2/r^2$. Consequently, when $r < r'$, $\psi_\lambda$ must be taken as (31) and when $r > r'$, as (30). Continuity of the logarithmic derivative determines $r'$:

$$\frac{|\lambda|}{r'} = \left(\frac{1}{2} - \sqrt{\alpha + \frac{1}{4}}\right)\frac{1}{r'\ln(r'/R_0)}\,. \tag{33}$$

Under these conditions, (29) becomes

$$\Delta E(\lambda) = 2\pi\lambda^2\left[\int_0^{r'} \psi_0^*(r)\psi_\lambda(r)\frac{\mathrm{d}r}{r} + \int_{r'}^{R} (\psi_0(r))^2\frac{\mathrm{d}r}{r}\right]. \tag{34}$$

After some algebra, we get

$$\Delta E(\lambda) \sim |\lambda|^{\sqrt{1+4\alpha}} \qquad 0 \leqslant \alpha < \tfrac{3}{4} \tag{35a}$$

$$\sim \lambda^2 |\ln|\lambda|| \qquad \alpha = \tfrac{3}{4} \tag{35b}$$

$$\sim \lambda^2 \qquad \alpha > \tfrac{3}{4}\,. \tag{35c}$$

In the case (35$a$), the variable $X = \theta/t^{1/\sqrt{1+4\alpha}}$ is distributed according to the Levy law

$$P(X) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\lambda\, \mathrm{e}^{-\mathrm{i}\lambda X}\, \mathrm{e}^{-|\lambda|^{\sqrt{1+4\alpha}}}\,. \tag{36}$$

In particular, all the moments are infinite (unless they trivially cancel).

Likewise, the law corresponding to (35$b$) has all its moments infinite. However, it is not known as a stable Lévy law. Case (35$c$) shows that when the potential is sufficiently repulsive at the origin, the winding angle distribution again becomes Gaussian: all the moments are finite. Thus a sharp transition occurs at the critical value $\alpha = \frac{3}{4}$.

We expect that result (35$a$) still holds for the range $-\frac{1}{4} < \alpha < 0$, corresponding to attractive potentials. Thus, the set of Lévy laws would be exhaustive. However, further investigation is needed to confirm this point.

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